

# EQUIVARIANT COMPACTIFICATIONS OF TWO-DIMENSIONAL ALGEBRAIC GROUPS

ULRICH DERENTHAL AND DANIEL LOUGHRAN

**ABSTRACT.** We classify generically transitive actions of semidirect products  $\mathbb{G}_a \rtimes \mathbb{G}_m$  on  $\mathbb{P}^2$ . Motivated by the program to study the distribution of rational points on del Pezzo surfaces (Manin's conjecture), we determine all (possibly singular) del Pezzo surfaces that are equivariant compactifications of homogeneous spaces for semidirect products  $\mathbb{G}_a \rtimes \mathbb{G}_m$ .

## CONTENTS

1. Introduction	1
2. Generalities on algebraic groups	3
3. Actions on the projective plane	7
4. Actions on generalised del Pezzo surfaces	11
References	18

## 1. INTRODUCTION

In this note, we are concerned with the classification of algebraic surfaces that are equivariant compactifications of two-dimensional connected linear algebraic groups. Over an algebraically closed field  $K$  of characteristic 0, any such group is isomorphic to the torus  $\mathbb{G}_m^2$ , the additive group  $\mathbb{G}_a^2$  or a semidirect product  $\mathbb{G}_a \rtimes \mathbb{G}_m$ .

Here, varieties admitting an action of a connected linear algebraic group  $G$  with an open dense orbit are called *equivariant compactifications of homogeneous spaces for  $G$* . If the stabiliser of a point in the open dense orbit is trivial, then we simply say that the variety is an *equivariant compactification of  $G$* .

Equivariant compactifications of tori are widely studied in toric geometry. The classification of equivariant compactification of additive groups  $\mathbb{G}_a^n$  was initiated by Hassett and Tschinkel [HT99]. Here, we start the classification of equivariant compactifications of semidirect products  $\mathbb{G}_a \rtimes \mathbb{G}_m$ . We focus on del Pezzo surfaces (possibly with rational double points) having such a structure.

This has arithmetic motivations. Namely, the distribution of rational points on Fano varieties over number fields is predicted by Manin's conjecture [BM90], giving a precise asymptotic formula for the number of rational points of bounded height. Using methods of harmonic analysis, it has been proved for toric varieties [BT98a], for equivariant compactifications of  $\mathbb{G}_a^n$

---

2010 *Mathematics Subject Classification.* 14L30 (14J26, 11D45).

[CLT02] and recently for certain equivariant compactifications of  $\mathbb{G}_a \rtimes \mathbb{G}_m$  [TT12].

Furthermore, Manin’s conjecture is studied systematically in dimension 2, where Fano varieties are del Pezzo surfaces, primarily using universal torsors combined with various analytic techniques. See [Bro09, Chapter 2] for an overview. In the version stated in [BT98b], Manin’s conjecture is expected to hold for any del Pezzo surface whose singularities are rational double points (i.e. canonical); different behaviour occurs if one allows other singularities (see [BT98b, Example 5.1.1]).

Therefore, it is important to know which del Pezzo surfaces with at most rational double points are equivariant compactifications so that they may be covered by the results from harmonic analysis. It turns out that this depends only on the *type* of a del Pezzo surface (which can be expressed by its degree, the types of its singularities in the **ADE**-classification and the number of its lines, where the latter is relevant only in a few cases).

Toric del Pezzo surfaces are easily identified; see [Der06b, Figure 1], for example. Del Pezzo surfaces that are equivariant compactifications of  $\mathbb{G}_a^2$  were classified in [DL10]. This leaves the classification of those del Pezzo surfaces that are equivariant compactifications of semidirect products  $\mathbb{G}_a \rtimes \mathbb{G}_m$ , which is the main theorem of this paper.

**Theorem 1.** *A del Pezzo surface, possibly singular with rational double points, of degree  $d$  is an equivariant compactification of some semidirect product  $\mathbb{G}_a \rtimes \mathbb{G}_m$  if and only if it has one of the following types:*

- $d \geq 7$ ,
- $d = 6$  of types  $\mathbf{A}_2 + \mathbf{A}_1$ ,  $\mathbf{A}_2$ ,  $2\mathbf{A}_1$ ,  $\mathbf{A}_1$  (with three or four lines),
- $d = 5$  of types  $\mathbf{A}_3$ ,  $\mathbf{A}_2 + \mathbf{A}_1$ ,  $\mathbf{A}_2$ ,
- $d = 4$  of types  $\mathbf{A}_3 + 2\mathbf{A}_1$ ,  $\mathbf{D}_4$ ,  $\mathbf{A}_3 + \mathbf{A}_1$ .

*Additionally, precisely the following types are equivariant compactifications of a homogeneous space for some semidirect product  $\mathbb{G}_a \rtimes \mathbb{G}_m$ :*

- $d = 5$  of type  $\mathbf{A}_4$ ,
- $d = 4$  of type  $\mathbf{D}_5$ ,  $\mathbf{A}_4$ ,
- $d = 3$  of type  $\mathbf{E}_6$ ,  $\mathbf{A}_5 + \mathbf{A}_1$ .

Theorem 1 is visualised diagrammatically in Figure 1. Note that as remarked in [HT99, Section 2], any toric variety has a unique structure as a toric variety. This may however fail for other algebraic groups. For example, even  $\mathbb{P}^n$  has infinitely many different structures as an equivariant compactification of  $\mathbb{G}_a^n$  for  $n \geq 6$  [HT99, Example 3.6]. We consider the corresponding problem for each semidirect product  $\mathbb{G}_a \rtimes \mathbb{G}_m$ . In the case where  $\mathbb{G}_a \rtimes \mathbb{G}_m$  is not the direct product  $\mathbb{G}_a \times \mathbb{G}_m$ , we show that up to equivalence  $\mathbb{P}^2$  admits precisely two different structures as an equivariant compactification of  $\mathbb{G}_a \rtimes \mathbb{G}_m$  (see Theorem 13). We also prove that it admits infinitely many different structures as an equivariant compactification of a homogeneous space for each  $\mathbb{G}_a \rtimes \mathbb{G}_m$ .

Note that a related result is proved in [AHHL12, Section 6]. There however, only the classification of those equivariant compactifications of homogeneous spaces (“almost homogeneous” in their terminology) having Picard

number one is considered, while our techniques allow us to identify the equivariant compactifications of  $\mathbb{G}_a \rtimes \mathbb{G}_m$ . Moreover, in Section 4 we also give results towards classifying the possible actions which may occur for the surfaces listed in Theorem 1, for example we show which stabilisers may arise.

The layout of this paper is as follows. In Section 2 we gather various facts on algebraic group actions and on equivariant compactifications of homogeneous spaces. In Section 3 we classify the different structures that  $\mathbb{P}^2$  admits as an equivariant compactification of a homogeneous space for each semidirect product  $\mathbb{G}_a \rtimes \mathbb{G}_m$ . Finally we finish off by considering del Pezzo surfaces and proving Theorem 1 in Section 4. Throughout this paper we work over an algebraically closed field  $K$  of characteristic zero and all algebraic groups will be linear.

**Acknowledgements:** The first-named author was supported by grant DE 1646/2-1 of the Deutsche Forschungsgemeinschaft and by the Center for Advanced Studies of LMU München. The majority of this work was completed whilst the second-named author was working at l’Institut de Mathématiques de Jussieu and supported by ANR PEPR. The authors would like to thank Ivan Arzhantsev and Pierre Le Boudec for their comments.

## 2. GENERALITIES ON ALGEBRAIC GROUPS

**2.1. Actions of algebraic groups.** We begin by collecting various results on actions of (always linear) algebraic groups on varieties.

**Definition 2.** Let  $G$  be a connected algebraic group and  $X$  a proper normal variety. If  $X$  admits an action of  $G$  that is *generically transitive* (i.e. transitive on some dense open subset), we say that  $X$  is an *equivariant compactification of a homogeneous space for  $G$* . If moreover the action is also *generically free* (i.e. free on some dense open subset), then we say that  $X$  is an *equivariant compactification of  $G$* .

For motivation with this terminology, suppose that  $X$  is an equivariant compactification of a homogeneous space for  $G$  and let  $H$  be the stabiliser of a general point (i.e. a point in the open dense orbit). Then  $X$  contains an open subset isomorphic to the homogeneous space  $G/H$  and the action of  $G$  on  $X$  extends the natural action of  $G$  on  $G/H$ . If moreover  $G/H$  is affine (this holds for example if  $H$  is finite, see the proof of [Bor91, Proposition 6.15]), then the complement of  $G/H$  in  $X$  is a divisor [Gro67, Corollaire 21.12.7], which we call the *boundary* of the action. As example, note that a toric variety is by definition an equivariant compactification of an algebraic torus. As algebraic tori are commutative however, every homogeneous space for a torus is in fact itself a torus, in particular every equivariant compactification of a homogeneous space for an algebraic torus is also a toric variety. To obtain homogeneous spaces that are not themselves algebraic groups, one needs to consider non-commutative groups; we shall see many such examples in Section 4.

We shall be interested in classifying generically transitive actions up to the following notion of equivalence.

**Definition 3.** Let  $G$  be an algebraic group acting on varieties  $X_1$  and  $X_2$ . Then an equivalence of (left)  $G$ -actions is a commutative diagram

$$\begin{array}{ccc} G \times X_1 & \xrightarrow{(\alpha, j)} & G \times X_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{j} & X_2 \end{array}$$

where  $\alpha : G \rightarrow G$  is an automorphism and  $j : X_1 \rightarrow X_2$  is an isomorphism.

Note that in order to classify generically transitive actions up to equivalence, we need only consider *left* actions. Indeed, if  $G$  acts on the right on a variety  $X$  via  $(x, g) \mapsto xg$ , then we obtain a left action of  $G$  on  $X$  defined by  $(g, x) \mapsto xg^{-1}$ . This left action is obviously generically transitive (resp. generically free) if and only if the original action is. Throughout this paper we will therefore assume that all groups act on the left. We begin with the following elementary result.

**Lemma 4.** *Let  $X$  be an equivariant compactification of a homogeneous space for an algebraic group  $G$  and suppose that  $\dim X = \dim G$ . Then the stabiliser of a general point is finite.*

*Proof.* Let  $H$  be the stabiliser of some point in the open dense orbit, which is a closed subgroup of  $G$ . Note that choosing a different point in this orbit gives a conjugate closed subgroup of  $G$ , in particular the cardinality of  $H$  is in fact independent of this choice. Also note that we have  $\dim G/H = \dim X$ , as  $X$  contains an open subset isomorphic to  $G/H$ . Therefore we see that the fibres of the quotient map  $G \rightarrow G/H$  have dimension zero and hence that  $H$  is finite.  $\square$

Next recall that given an action of an algebraic group  $G$  on a variety  $X$  and a line bundle  $L$  on  $X$ , a  $G$ -linearisation of  $L$  is an action of  $G$  on  $L$  that respects the action of  $G$  on  $X$  (see [Dol03, Chapter 7]).

**Lemma 5.** *Let  $G$  be a connected algebraic group such that  $\text{Pic}(G) = 0$ , and suppose that  $G$  acts on some normal variety  $X$ . Then every line bundle on  $X$  admits a  $G$ -linearisation.*

*In particular for any  $n \in \mathbb{N}$ , every projective representation  $G \rightarrow \text{PGL}_n$  admits a lift to a representation  $G \rightarrow \text{GL}_n$ , i.e. there exists a homomorphism  $G \rightarrow \text{GL}_n$  such that the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\quad} & \text{GL}_n \\ & \searrow & \downarrow \\ & & \text{PGL}_n \end{array}$$

*is commutative.*

*Proof.* By [Dol03, Theorem 7.2], as  $G$  is connected we have an exact sequence

$$\text{Pic}^G(X) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(G),$$

where  $\text{Pic}^G(X)$  denotes the group of isomorphism classes of  $G$ -linearised line bundles on  $X$ . As  $\text{Pic}(G) = 0$ , the map  $\text{Pic}^G(X) \rightarrow \text{Pic}(X)$  is surjective and hence every line bundle on  $X$  admits a  $G$ -linearisation.

To prove the second part of the lemma, note that a projective representation  $G \rightarrow \mathrm{PGL}_n$  gives rise to an action of  $G$  on  $\mathbb{P}^{n-1}$ . By the first part of the lemma, the line bundle  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$  admits a  $G$ -linearisation. Therefore we obtain an action on the  $n$ -dimensional vector space  $H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ , which is the required lift to a representation  $G \rightarrow \mathrm{GL}_n$ .  $\square$

The algebraic groups of primary interest in this paper (namely  $\mathbb{G}_a, \mathbb{G}_m$  and semidirect products  $\mathbb{G}_a \rtimes \mathbb{G}_m$ ) all have trivial Picard groups [Dol03, Remark 7.3]. Note also that in general, the choice of linearisation will not be unique if  $G$  admits non-trivial characters (see [Dol03, (7.3)]).

Next, we obtain a criterion to help determine whether certain morphisms to projective space are equivariant.

**Lemma 6.** *Let  $X$  be a normal variety together with the action of an algebraic group  $G$ . Let  $L$  be a line bundle on  $X$  that is generated by its global sections such that  $W = H^0(X, L)$  is finite dimensional and which admits a  $G$ -linearisation. Let also  $s_0, \dots, s_n$  be linearly independent sections that generate a base-point free linear series  $V \subset W$ . Then if  $\varphi : X \rightarrow \mathbb{P}(V)$  denotes the associated morphism, the following are equivalent.*

- (1)  $V \subset W$  is invariant under the action of  $G$ .
- (2) The composed morphism  $X \rightarrow \mathbb{P}(V) \subset \mathbb{P}(W)$  is  $G$ -equivariant.
- (3)  $\mathbb{P}(V) \subset \mathbb{P}(W)$  is invariant under the action of  $G$ .

*Proof.* The proof that (1) implies (2) can be found in [Dol03, Section 7.3]. To show that (2) implies (3), first note that if we let  $Y = \varphi(X)$ , then  $\mathbb{P}(V)$  is the only linear subspace of  $\mathbb{P}(W)$  of dimension  $n$  that contains  $Y$ . Indeed, suppose that there exists another linear subspace  $H \subset \mathbb{P}(W)$  of dimension  $n$  that contains  $Y$ . Then  $H \cap \mathbb{P}(V)$  is a linear subspace of dimension  $\leq n-1$  that contains  $Y$ . In particular, this implies that there is a linear relation between  $s_0, \dots, s_n$ . However, this gives a contradiction as  $s_0, \dots, s_n$  were chosen to be linearly independent. Next, the  $G$ -equivariance of  $\varphi$  implies that  $g\mathbb{P}(V)$  is a linear subspace of  $\mathbb{P}(W)$  of dimension  $n$  that contains  $Y$ , for all  $g \in G$ . Therefore we have  $g\mathbb{P}(V) = \mathbb{P}(V)$ , i.e.  $\mathbb{P}(V) \subset \mathbb{P}(W)$  is invariant under the action of  $G$ . This proves (3).

Finally, we show that (3) implies (1). The fact that  $\mathbb{P}(V) \subset \mathbb{P}(W)$  is invariant under the action of  $G$  implies that for any line  $E \subset V$  we have  $gE \subset V$  for all  $g \in G$ . Applying this to the line spanned by each  $s \in V$ , we deduce that  $gs \in V$  for all  $g \in G$ , which proves (1).  $\square$

Note that as the property of  $V \subset W$  being invariant under the action of  $G$  is independent of the choice of basis, we see that (2) in Lemma 6 is in fact independent of the choice of  $s_0, \dots, s_n$ . Also, as (3) is independent of the choice of  $G$ -linearisation on  $L$ , we see that (1) is also independent of the choice of  $G$ -linearisation.

We next consider how the property of being an equivariant compactification of a homogeneous space behaves with respect to birational morphisms.

**Lemma 7.** *Let  $G$  be a connected algebraic group and let  $X$  be an equivariant compactification of a homogeneous space for  $G$ . Let  $\pi : \tilde{X} \rightarrow X$  be the blow up of  $X$  at a subvariety  $V \subset X$  that is invariant under the action of  $G$ .*

Then  $\tilde{X}$  is an equivariant compactification of a homogeneous space for  $G$  in such a way that  $\pi$  is a  $G$ -equivariant morphism.

*Proof.* From the universal property of blow-ups [Har77, Corollary II.7.15], we obtain a morphism  $G \times \tilde{X} \rightarrow \tilde{X}$ . It is easy to see that this gives the required action (see the proof of [DL10, Lemma 3]).  $\square$

**Lemma 8.** *Let  $G$  be a connected algebraic group and let  $X$  be a smooth equivariant compactification of a homogeneous space for  $G$ . Let  $\pi : X \rightarrow Y$  be a birational morphism to a normal projective variety  $Y$ . Then  $Y$  is an equivariant compactification of a homogeneous space for  $G$  in such a way that  $\pi$  is a  $G$ -equivariant morphism.*

*Proof.* For equivariant compactifications of  $G$ , see [TT12, Proposition 1.3]. The exact same proof works for equivariant compactifications of homogeneous spaces for  $G$ , as the fact that the stabiliser of a general point is trivial is not used in the proof.  $\square$

Combining these results we obtain the following.

**Proposition 9.** *Let  $G$  be a connected algebraic group,  $S$  a singular projective normal surface and let  $\pi : \tilde{S} \rightarrow S$  be a minimal desingularisation. Then  $S$  is an equivariant compactification of a homogeneous space for  $G$  if and only if  $\tilde{S}$  is, and in which case  $\pi$  is a  $G$ -equivariant morphism.*

*Proof.* The proof of this lemma is essentially the same as the proof of [DL10, Lemma 4]. The fact that  $S$  is normal implies that the singular locus consists of a finite set of singularities. As  $G$  is connected, each of these singularities must be fixed under the action of  $G$ . Since the map  $\pi$  is given by successively blowing up these singularities, on applying Lemma 7 and Lemma 8 we deduce the result.  $\square$

Note that as the  $G$ -equivariant morphisms in Lemma 7, Lemma 8 and Proposition 9 are birational, they will preserve the order of the stabiliser of each point in the open dense orbit.

**2.2. Semidirect products  $\mathbb{G}_a \rtimes \mathbb{G}_m$ .** We now turn our attention to semidirect products of  $\mathbb{G}_a$  and  $\mathbb{G}_m$ . Note that one may write down all such groups in a fairly simple way. Namely, a semidirect product  $\mathbb{G}_a \rtimes \mathbb{G}_m$  is given by a homomorphism  $\mathbb{G}_m \rightarrow \text{Aut}(\mathbb{G}_a) = \mathbb{G}_m$ . Since homomorphisms  $\mathbb{G}_m \rightarrow \mathbb{G}_m$  are given by  $t \mapsto t^d$  for any integer  $d$ , any such semidirect product has the form  $G_d = \mathbb{G}_a \rtimes_{\phi_d} \mathbb{G}_m$  with  $\phi_d(t)(b) = t^d b$ . The group law on  $G_d$  is given by

$$(b, t) \cdot (b', t') = (b + t^d b', tt').$$

We keep this notation throughout this paper. Note that we have obvious isomorphisms  $G_d \cong G_{-d}$  and  $G_0 \cong \mathbb{G}_a \times \mathbb{G}_m$ .

Later on, we will want to have some information about stabilisers of generically transitive actions. For this it will be useful to know which finite subgroups can occur.

**Lemma 10.** *Any finite subgroup of  $G_d$  is conjugate to one of the form*

$$\mu_n \rightarrow G_d, \quad \zeta \mapsto (0, \zeta),$$

for some  $n \in \mathbb{N}$ . Such a subgroup is normal if and only if  $n \mid d$ , in which case  $G_d/\mu_n \cong G_{d/n}$ .

*Proof.* Let  $H \subset G_d$  be a finite subgroup. Restricting the exact sequence

$$0 \rightarrow \mathbb{G}_a \rightarrow G_d \rightarrow \mathbb{G}_m \rightarrow 1,$$

to  $H$ , we see that  $H$  injects into  $\mathbb{G}_m$ . Indeed as  $K$  has characteristic zero  $\mathbb{G}_a$  has no non-trivial finite subgroups and hence  $H \cap \mathbb{G}_a = 0$ . Therefore, there exists  $n \in \mathbb{N}$  such that  $H \cong \mu_n$  as an algebraic group, in particular  $H$  is cyclic and generated by a semisimple element. Such an element is conjugate to one in the maximal torus  $T = \{(0, t) : t \in \mathbb{G}_m\}$  by [Bor91, Theorem III.10.6]. This completes the proof of the first part of the lemma.

A simple calculation shows that  $\mu_n$  is not normal if  $n \nmid d$ . If  $n \mid d$ , then the map

$$G_d \rightarrow G_{d/n}, \quad (b, t) \mapsto (b, t^n),$$

has kernel  $\mu_n$  and gives the required isomorphism.  $\square$

Note that it follows from Lemma 10 that if we wish to classify generically transitive actions of  $G_d$  on a certain surface  $S$  for every  $d \in \mathbb{Z}$ , we may reduce to the case where the action is *faithful*. Indeed the kernel of a generically transitive action of  $G_d$  on  $S$  will be a finite normal subgroup by Lemma 4, hence quotienting out we obtain a faithful generically transitive action of  $G_{d/n}$  on  $S$  for some  $n \mid d$ .

### 3. ACTIONS ON THE PROJECTIVE PLANE

We now classify the generically transitive actions of  $G_d$  on  $\mathbb{P}^2$ . We begin with a lemma on three-dimensional representations of  $\mathbb{G}_a$ .

**Lemma 11.** *Let  $f : \mathbb{G}_a \rightarrow \mathrm{GL}_3$  be a faithful representation whose image consists only of upper triangular matrices. Then there exist  $\alpha_1, \alpha_2, \alpha_3 \in K$  not all zero such that*

$$f(b) = \begin{pmatrix} 1 & \alpha_1 b & \alpha_2 b + \frac{\alpha_1 \alpha_3}{2} b^2 \\ 0 & 1 & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* By assumption, we may assume that

$$f(b) = \begin{pmatrix} f_{1,1}(b) & f_{1,2}(b) & f_{1,3}(b) \\ 0 & f_{2,2}(b) & f_{2,3}(b) \\ 0 & 0 & f_{3,3}(b) \end{pmatrix},$$

where all the  $f_{i,j}(b)$  are polynomial expressions in  $b$ . For this to define an action we must have

$$f_{i,i}(b) \cdot f_{i,i}(b') = f_{i,i}(b + b'), \quad (1)$$

for  $i = 1, 2, 3$ , i.e. each  $f_{i,i}$  defines a homomorphism  $f_{i,i} : \mathbb{G}_a \rightarrow \mathbb{G}_m$ . Such a homomorphism must be trivial, hence we have  $f_{i,i}(b) = 1$  for each  $b \in \mathbb{G}_a$  and  $i = 1, 2, 3$ .

Next, differentiating the map  $f$  gives an injection of Lie algebras  $df : \mathfrak{g} \rightarrow \mathfrak{gl}_3$ , where  $\mathfrak{g}$  denotes the Lie algebra of  $\mathbb{G}_a$ . The morphism  $df$  sends a generator of  $\mathfrak{g}$  to a nilpotent matrix

$$\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & \alpha_3 \\ 0 & 0 & 0 \end{pmatrix},$$

where  $\alpha_1, \alpha_2, \alpha_3 \in K$ , at least one of which is non-zero. On exponentiating this map we obtain the result.  $\square$

The following lemma is the key step in the classification of the generically transitive actions on  $\mathbb{P}^2$  up to equivalence. It will also be used later on in our study of such actions on generalised del Pezzo surfaces.

**Lemma 12.** *Let  $d \in \mathbb{Z}$  and let  $\rho : G_d \rightarrow \mathrm{PGL}_3$  be a faithful representation whose image consists of only upper triangular matrices. Then there exists an element  $g \in G_d$  and  $k_1, k_2 \in \mathbb{Z}$  not both zero and  $\alpha_1, \alpha_2, \alpha_3 \in K$  not all zero such that*

$$\rho(g^{-1}(b, t)g) = \begin{pmatrix} t^{k_1} & \alpha_1 b t^{k_2} & \alpha_2 b + \frac{\alpha_1 \alpha_3}{2} b^2 \\ 0 & t^{k_2} & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the following four conditions must hold:

- $\alpha_1 = 0$  or  $k_1 = k_2 + d$ ,
- $\alpha_2 = 0$  or  $k_1 = d$ ,
- $\alpha_3 = 0$  or  $k_2 = d$ ,
- $\alpha_1 \alpha_2 \alpha_3 = 0$ .

*Proof.* Let  $B = \{(b, 1) : b \in \mathbb{G}_a\}$  denote the normal subgroup of  $G_d$  isomorphic to  $\mathbb{G}_a$ , and let  $T$  denote the maximal torus  $T = \{(0, t) : t \in \mathbb{G}_m\}$ . The first step of the proof is to analyse the behaviour of  $\rho$  when restricted to  $B$  and  $T$ . Note that by Lemma 5, there exists a lift of  $\rho$  to a faithful representation  $f : G_d \rightarrow \mathrm{GL}_3$  that will take the form

$$\begin{pmatrix} f_{1,1}(b, t) & f_{1,2}(b, t) & f_{1,3}(b, t) \\ 0 & f_{2,2}(b, t) & f_{2,3}(b, t) \\ 0 & 0 & f_{3,3}(b, t) \end{pmatrix},$$

where all the  $f_{i,j}(b, t)$  are polynomial expressions in  $b, t, t^{-1}$ . For this to define an action, the following relations must hold

$$f_{i,i}(b, t) \cdot f_{i,i}(b', t') = f_{i,i}(b + t^d b', tt'), \quad (2)$$

for  $i = 1, 2, 3$ . Applying Lemma 11 we see that  $f_{i,i}(b, 0) = 1$ . Therefore it follows from (2) that each  $f_{i,i}$  defines a homomorphism  $f_{i,i} : T \rightarrow \mathbb{G}_m$ , so we must have  $f_{i,i}(b, t) = f_{i,i}(0, t) = t^{k_i}$  for some  $k_i \in \mathbb{Z}$  and  $i = 1, 2, 3$ . Note that we may obviously choose the lift  $f$  so that  $k_3 = 0$ . Moreover, we claim that at least one of  $k_1$  and  $k_2$  is non-zero. Indeed, otherwise  $f$  restricted to  $T$  would give a map  $T \rightarrow \mathrm{GL}_3$  whose image is unipotent. As  $T \cong \mathbb{G}_m$ , such a map must be trivial, which contradicts the fact that  $f$  is faithful.

Next we find a maximal torus of  $G_d$  that has diagonal image under  $f$ . Let  $D_3 \subset \mathrm{GL}_3$  denote the subgroup of diagonal matrices and let  $H = D_3 \cap f(G_d)$ , which is a closed algebraic subgroup of both  $D_3$  and  $f(G_d)$ . Since



one of the  $k_i$  is non-zero, we see that  $H$  is not finite. Thus if we let  $H^0$  denote the connected component of the identity of  $H$ , it follows that  $H^0$  is an algebraic torus of dimension one as it is a connected one-dimensional algebraic subgroup of  $D_3 \cong \mathbb{G}_m^3$ . So  $H^0$  defines a maximal torus in  $f(G_d)$ , and pulling back via  $f$  we obtain a maximal torus in  $G_d$  with diagonal image. However as any two maximal tori are conjugate (see e.g. [Bor91, Theorem III.10.6]), there exists an element  $g \in G_d$  such that  $f(g^{-1}Tg)$  consists of diagonal matrices. Moreover by the above we may assume that  $f(g^{-1}(0, t)g) = \text{diag}(t^{k_1}, t^{k_2}, 1)$ .

Next note that the map  $b \mapsto f(g^{-1}(b, 1)g)$  is a faithful representation of  $\mathbb{G}_a$  that consists of upper triangular matrices. Hence applying Lemma 11 and using the fact that  $f(g^{-1}(b, t)g) = f(g^{-1}(b, 1)g)f(g^{-1}(0, t)g)$ , we see that there exist  $\alpha_1, \alpha_2, \alpha_3 \in K$  not all zero such that  $f(g^{-1}(b, t)g)$  is given by

$$\begin{pmatrix} t^{k_1} & \alpha_1 b t^{k_2} & \alpha_2 b + \frac{\alpha_1 \alpha_3}{2} b^2 \\ 0 & t^{k_2} & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

One can check that this defines a homomorphism if and only if

$$\alpha_1(t^{k_1} - t^{d+k_2}) = \alpha_2(t^{k_1} - t^d) = \alpha_3(t^{k_2} - t^d) = 0, \quad (3)$$

for all  $t \in K^*$ . This gives the list of conditions in the lemma. To finish the proof, it suffices to note that if  $\alpha_1 \alpha_2 \alpha_3 \neq 0$  then (3) implies that  $k_1 = k_2 = d = 0$ , which does not give a faithful representation.  $\square$

We are now ready to classify the faithful generically transitive actions of  $G_d$  on  $\mathbb{P}^2$ . We first define the actions that we will be interested in. Let  $d \in \mathbb{Z}$  and let  $k \in \mathbb{Z} \setminus 0$ . We define a generically transitive action of  $G_d$  on  $\mathbb{P}^2$  by

$$\tau_{d,k}(b, t) = \begin{pmatrix} t^k & 0 & 0 \\ 0 & t^d & b \\ 0 & 0 & 1 \end{pmatrix}.$$

The following facts are easy to check.

- The stabiliser of a general point has order  $|k|$ .
- The representation is faithful if and only if  $\gcd(|k|, |d|) = 1$ .
- The boundary divisor consists of the two lines  $\{x = 0\}$  and  $\{z = 0\}$ .
- If  $k \neq d$ , the only fixed points are  $(1 : 0 : 0)$  and  $(0 : 1 : 0)$ . If  $k = d$ , then the fixed points are exactly the points on the line  $z = 0$ .

Note that  $\tau_{d,k}$  is not equivalent to  $\tau_{d,k'}$  for any  $|k| \neq |k'|$ , as the stabilisers of a general point are different in each case. Also  $\tau_{d,k}$  is not equivalent to  $\tau_{d,-k}$  for  $d \neq 0$  as these have inequivalent action on the line  $\{z = 0\}$ . One sees easily however that  $\tau_{0,k}$  is equivalent to  $\tau_{0,-k}$  on applying the automorphism  $(b, t) \mapsto (b, t^{-1})$  of  $G_0 = \mathbb{G}_a \times \mathbb{G}_m$ . We also have another faithful generically transitive action of  $G_d$  on  $\mathbb{P}^2$  given by

$$\rho_d(b, t) = \begin{pmatrix} t^{2d} & b t^d & b^2/2 \\ 0 & t^d & b \\ 0 & 0 & 1 \end{pmatrix},$$

for any  $d \neq 0$ . Here again it is easy to check the following.

- The stabiliser of a general point has order  $2|d|$ .

- The boundary divisor consists of the line  $\{z = 0\}$  and the conic  $\{y^2 = 2xz\}$ .
- The only fixed point is  $(1 : 0 : 0)$ .

Note that the boundary divisor for  $\rho_d$  does not have strict normal crossings as the conic lies tangent to the line. Also, it is easy to see that  $\rho_d$  is not equivalent to  $\tau_{d,k}$  for any  $kd \neq 0$ , as there is no automorphism of  $\mathbb{P}^2$  that swaps a line and a conic. Our main theorem in this section is that any faithful generically transitive action of  $G_d$  of  $\mathbb{P}^2$  is of the above form, up to equivalence.

**Theorem 13.** *Let  $d \neq 0$ . Any faithful generically transitive action of  $G_d$  on  $\mathbb{P}^2$  is equivalent to either  $\tau_{d,k}$  for some  $k \neq 0$  with  $\gcd(|k|, |d|) = 1$  or  $\rho_d$ . Any faithful generically transitive action of  $G_0$  on  $\mathbb{P}^2$  is equivalent to  $\tau_{0,k}$  for some  $k \in \mathbb{N}$ .*

*Proof.* The action of  $G_d$  on  $\mathbb{P}^2$  gives rise to a faithful representation  $\rho : G_d \rightarrow \mathrm{PGL}_3$ . As  $G_d$  is solvable, it follows from the Lie-Kochin theorem [Bor91, Corollary III.10.5] (applied to a lift of  $\rho$  obtained via Lemma 5), that we may conjugate by an element of  $\mathrm{PGL}_3$  to obtain an equivalent action whose image consists of only upper triangular matrices. This corresponds to the fact that the action on  $\mathbb{P}^2$  leaves  $(1 : 0 : 0)$  and  $\{z = 0\}$  invariant. Therefore applying Lemma 12, we see that up to equivalence  $\rho(b, t)$  takes the form

$$\begin{pmatrix} t^{k_1} & \alpha_1 b t^{k_2} & \alpha_2 b + \frac{\alpha_1 \alpha_3}{2} b^2 \\ 0 & t^{k_2} & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

We now proceed by considering the various possibilities on the  $\alpha_i$  given by Lemma 12. First, if  $\alpha_1 \alpha_2 \neq 0$  then we see that  $k_2 = 0, k_1 = d$  and  $\alpha_3 = 0$ . This action is not generically transitive for any  $d$ ; indeed it preserves the lines  $y = \lambda z$  for any  $\lambda \in K$ .

Next consider the case where  $\alpha_1 \alpha_3 \neq 0$  and hence  $\alpha_2 = 0$ . Then Lemma 12 implies that  $k_1 = 2d$  and  $k_2 = d$  and therefore  $d \neq 0$ . We claim that this action is equivalent to  $\rho_d$ . Indeed, the conic  $\{\alpha_1 y^2 = 2\alpha_3 xz\}$  is invariant under the action. The automorphism of  $\mathbb{P}^2$  given by  $x \mapsto \alpha_3 x / \alpha_1$  moves this conic to the conic  $\{y^2 = 2xz\}$  and gives rise to an equivalent action given by

$$\begin{pmatrix} t^{2d} & \alpha_3 b t^d & (\alpha_3 b)^2 / 2 \\ 0 & t^d & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

On performing the automorphism  $(b, t) \mapsto (b/\alpha_3, t)$  of  $G_d$ , which rescales  $b$ , we obtain  $\rho_d$ . Thus we may assume that  $\alpha_1 \alpha_3 = 0$  and that  $\rho$  takes the form

$$\begin{pmatrix} t^{k_1} & \alpha_1 b t^{k_2} & \alpha_2 b \\ 0 & t^{k_2} & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

If  $\alpha_2 \alpha_3 \neq 0$  then Lemma 12 tell us  $k_1 = k_2 = d$  and  $\alpha_1 = 0$ . Clearly this action is not faithful unless  $d = 1$ , in which case it gives

$$\begin{pmatrix} t & 0 & \alpha_2 b \\ 0 & t & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

The boundary here consists on the lines  $\{z = 0\}$  and  $\{\alpha_2 y = \alpha_3 x\}$ . Hence as before, we may perform an automorphism of  $\mathbb{P}^2$  that moves the line  $\{\alpha_2 y = \alpha_3 x\}$  to the line  $\{x = 0\}$  to obtain an action equivalent to  $\tau_{1,1}$ .

Thus we have reduced to the case where  $\alpha_1 \alpha_2 = \alpha_1 \alpha_3 = \alpha_2 \alpha_3 = 0$ . In particular only one of the  $\alpha_i$  can be non-zero, and we may even assume that  $\alpha_i = 1$  since applying the automorphism  $(b, t) \mapsto (b/\alpha_i, t)$  of  $G_d$  gives an equivalent action. This leaves the following three cases

$$\begin{pmatrix} t^k & 0 & 0 \\ 0 & t^d & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} t^d & 0 & b \\ 0 & t^k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} t^{d+k} & bt^k & 0 \\ 0 & t^k & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first action is  $\tau_{d,k}$  by definition, whereas the second action is seen to be equivalent to  $\tau_{d,k}$  on performing the automorphism of  $\mathbb{P}^2$  that swaps  $x$  and  $y$ . As for the third one, we notice that in  $\mathrm{PGL}_3$  we have

$$\begin{pmatrix} t^{d+k} & bt^k & 0 \\ 0 & t^k & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t^d & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-k} \end{pmatrix},$$

which is easily seen to be equivalent to  $\tau_{d,-k}$  on performing the automorphism of  $\mathbb{P}^2$  that swaps  $y$  and  $z$ .  $\square$

#### 4. ACTIONS ON GENERALISED DEL PEZZO SURFACES

**4.1. Recap on del Pezzo surfaces.** We now recall various facts that we will need on del Pezzo surfaces, which can be found for example in [DP80], [Man86] or [CT88]. As before, we work over an algebraically closed field  $K$  of characteristic 0.

A *generalised del Pezzo surface*  $\tilde{S}$  is a non-singular projective surface whose anticanonical class  $-K_{\tilde{S}}$  is big and nef. A normal projective surface  $S$  with ample anticanonical class  $-K_S$  is called an *ordinary del Pezzo surface* if it is non-singular and *singular del Pezzo surface* if its singularities are rational double points. Ordinary del Pezzo surfaces and minimal desingularisations of singular del Pezzo surfaces are generalised del Pezzo surfaces, and conversely every generalised del Pezzo surface arises in this way (see [CT88, Proposition 0.6]).

The *degree* of a generalised del Pezzo surface  $\tilde{S}$  is the self-intersection number  $(-K_{\tilde{S}}, -K_{\tilde{S}})$  of its anticanonical class. The degree of a singular del Pezzo surface  $S$  is defined to be the degree of its minimal desingularisation. For  $n \in \mathbb{N}$ , a  $(-n)$ -curve (or simply a *negative curve*) on a non-singular projective surface is a rational curve with self-intersection number  $-n$ . On generalised del Pezzo surfaces, only  $(-1)$ - or  $(-2)$ -curves may occur (see [CT88, page 29]). Moreover, a generalised del Pezzo surface is ordinary if and only if it contains no  $(-2)$ -curves.

A theorem of Demazure (see [CT88, Proposition 0.4]) says that any generalised del Pezzo surface  $\tilde{S}$  is isomorphic to either  $\mathbb{P}^2$  (degree 9),  $\mathbb{P}^1 \times \mathbb{P}^1$ , the Hirzebruch surface  $\mathbb{F}_2$  (both of degree 8) or is obtained from  $\mathbb{P}^2$  by a sequence

$$\tilde{S} = \tilde{S}_r \xrightarrow{\rho_r} \tilde{S}_{r-1} \rightarrow \cdots \rightarrow \tilde{S}_1 \xrightarrow{\rho_1} \tilde{S}_0 = \mathbb{P}^2$$

of  $r \leq 8$  blow-ups  $\rho_i : \tilde{S}_i \rightarrow \tilde{S}_{i-1}$  of points  $p_i \in \tilde{S}_{i-1}$  not lying on a  $(-2)$ -curve on  $\tilde{S}_{i-1}$ , for  $i = 1, \dots, r$ . The Picard group  $\text{Pic}(\tilde{S})$  of a generalised del Pezzo surface of degree  $d$  is a torsion-free abelian group of rank  $10 - d$ . For a generalised del Pezzo surface  $\tilde{S}$  of degree  $d \geq 3$ , the anticanonical linear system defines a birational morphism  $\pi : \tilde{S} \rightarrow S \subset \mathbb{P}^d$  to a surface  $S$ . If  $\tilde{S}$  is ordinary, then  $\pi$  is in fact a closed immersion. Otherwise,  $\pi$  contracts precisely the  $(-2)$ -curves on  $\tilde{S}$  to the singularities of  $S$ , and  $S$  is a singular del Pezzo surface with minimal desingularisation  $\tilde{S}$ .

The singularity type of a singular del Pezzo surface  $S$  is defined to be the dual graph of the configuration of  $(-2)$ -curves on the minimal desingularisation  $\tilde{S}$ . These graphs are always Dynkin diagrams and are labelled by (sums of)  $\mathbf{A}_n$  for  $n \geq 1$ ,  $\mathbf{D}_n$  for  $n \geq 4$ ,  $\mathbf{E}_6$ ,  $\mathbf{E}_7$ ,  $\mathbf{E}_8$ . Moreover in each degree, there are only finitely many possibilities for the configurations of the negative curves that may occur on generalised del Pezzo surfaces; these *types* can be distinguished by the **ADE**-types of the Dynkin diagrams for the  $(-2)$ -curves and the number of lines. The latter can be left out in most cases; exceptions are two  $\mathbf{A}_3$ -types (with four resp. five lines) in degree 4 and two  $\mathbf{A}_1$ -types (with three resp. four lines) in degree 6; all other exceptions have degree 1 and 2 and are not relevant for us. For some types there are infinitely many isomorphism classes, but for all types that turn out to be equivariant compactifications of homogeneous spaces for  $\mathbb{G}_a \rtimes \mathbb{G}_m$ , we will see that there is precisely one such surface up to isomorphism.

**4.2. Actions on generalised del Pezzo surfaces.** We now consider the classification of those generalised del Pezzo surfaces that admit a generically transitive action of  $G_d$  for some  $d$ , with the aim of proving Theorem 1. It turns out that such surfaces must satisfy a special geometric condition.

**Lemma 14.** *Let  $\tilde{S}$  be a generalised del Pezzo surface that is an equivariant compactification of a homogeneous space for  $G_d$  for some  $d$ . Then*

$$\#\{\text{negative curves on } \tilde{S}\} \leq \text{rk Pic } \tilde{S} + 1.$$

*Proof.* First note that if  $\tilde{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$  or  $\tilde{S} \cong \mathbb{F}_2$ , then there is at most one negative curve and the inequality trivially holds. So we may assume that  $\tilde{S}$  is obtained from  $\mathbb{P}^2$  by a sequence of  $r$  blow-ups. To prove the inequality in this case, it suffices to show that the boundary of the action consists of  $r + 2 = \text{rk Pic } \tilde{S} + 1$  irreducible curves. Indeed, let  $E$  be a negative curve on  $\tilde{S}$ . By Lemma 5, the line bundle  $\mathcal{O}_{\tilde{S}}(E)$  admits a  $G_d$ -linearisation, in particular the divisor class of  $E$  is invariant under the action of  $G_d$ . As  $E$  is the unique effective curve in its divisor class, we see that  $E$  itself is invariant under the action of  $G_d$  and therefore  $E$  must lie on the boundary. The fact that the boundary consists of  $r + 2$  irreducible curves then gives the required inequality.

To prove the claim we proceed by induction. Let  $X$  be a smooth projective equivariant compactification of a homogeneous space for  $G_d$  that contains a  $(-1)$ -curve  $E$  and let  $\pi : X \rightarrow Y$  be the map given by contracting  $E$ . Note that we may assume that  $Y$  is an equivariant compactification of a homogeneous space for  $G_d$  and that  $\pi$  is  $G_d$ -equivariant by Lemma 8. As  $\pi$  is an isomorphism outside  $E$ , we see that  $X$  has exactly one more boundary

component than  $Y$ . Applying this inductively to  $\tilde{S}$ , we see that the boundary of the action on  $\tilde{S}$  consists of  $r+n$  irreducible curves, where  $n$  is the number of irreducible curves on the boundary of the action on  $\mathbb{P}^2$ . However, by the classification given in Theorem 13 we know that  $n = 2$ . This proves the claim and hence completes the proof of the lemma.  $\square$

From the classification of generalised del Pezzo surfaces that can be found in [Der06b], for example, it is straightforward to write down the list of surfaces that satisfy the condition of Lemma 14. These are shown in Figure 1.

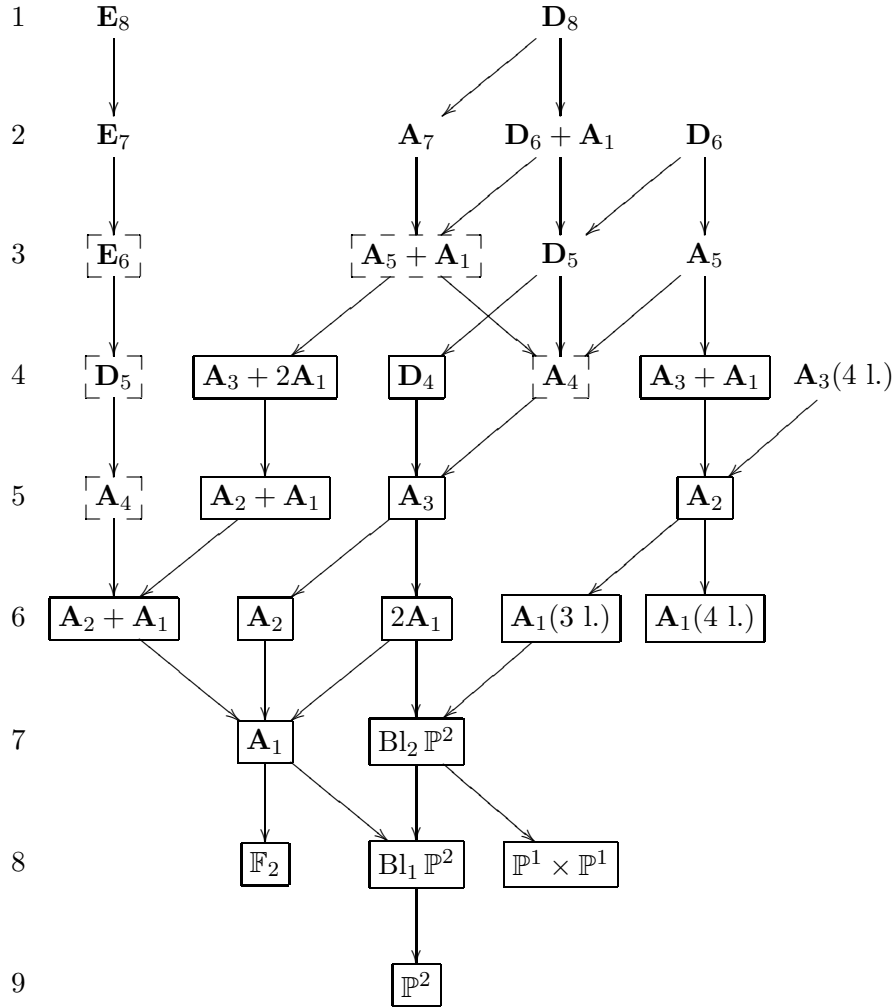


FIGURE 1. Generalised del Pezzo surfaces  $\tilde{S}$  in increasing degree with  $\#\{\text{negative curves on } \tilde{S}\} \leq \text{rk Pic}(\tilde{S}) + 1$ . The boxed ones are exactly the equivariant compactifications of  $G_d$  for some  $d$ . The dashed ones are exactly the equivariant compactifications of a homogeneous space for  $G_d$  for some  $d$ . Arrows denote blow-up maps (in degree  $\geq 4$ , only maps used in our proofs are included).

We note that for each of the types of degree *at most three* given in Figure 1, there is a *unique* surface over  $K$  with that type, up to isomorphism. Indeed, for the surfaces of degree three this follows from the classification given in [BW79]. This also implies uniqueness for all surfaces of degree greater than three, except for perhaps the quartic del Pezzo surface of type  $\mathbf{A}_3$  with four lines. However, we also have uniqueness in this case on noticing that such a surface is obtained by contracting a unique  $(-1)$ -curve on a del Pezzo surface of degree three and type  $\mathbf{A}_4$ . There is again a unique surface of this type by [BW79]. Note that this result does not hold for some of the lower degree surfaces in Figure 1, for example there are infinitely many generalised del Pezzo surfaces of degree two and type  $\mathbf{D}_6$  up to isomorphism (see [Ye02, Theorem 5.7]).

Next, it follows from Lemma 8 that we need only consider the “extremal” surfaces in Figure 1, namely if a surface is an equivariant compactification of (a homogeneous space for) some  $G_d$ , then so is any surface that lies below it in Figure 1. Conversely, if a surface is not an equivariant compactification of (a homogeneous space for)  $G_d$ , then no surface in Figure 1 that lies above it is either.

We now proceed to classify the generically transitive actions of  $G_d$  on some of the surfaces in Figure 1 up to equivalence, for each  $d \in \mathbb{Z}$ . We briefly outline the method that we shall use. Suppose that  $\rho : \tilde{S} \rightarrow \mathbb{P}^2$  is the composition of  $r \leq 6$  blow-ups of  $\mathbb{P}^2$  and that  $\tilde{S}$  admits a generically transitive action of  $G_d$  for some  $d$ . Then by Lemma 8, we obtain a generically transitive action of  $G_d$  on  $\mathbb{P}^2$  in such a way that  $\rho$  is  $G_d$ -equivariant. Also in every case we shall consider, we will be able to choose  $\rho$  in such a way that the line  $\{z = 0\}$  and the point  $(1 : 0 : 0)$  are images of negative curves on  $\tilde{S}$ . As the negative curves on  $\tilde{S}$  are invariant under the action (see the proof of Lemma 14), the line  $\{z = 0\}$  and the point  $(1 : 0 : 0)$  must also be invariant under the action on  $\mathbb{P}^2$ , and hence the action will have the form given by Lemma 12.

Therefore, we are reduced to the following question: for which of the actions given in Lemma 12 is the map  $\rho$   $G_d$ -equivariant? This is equivalent to asking whether the inverse of  $\rho$  is an  $G_d$ -equivariant birational map  $\rho^{-1} : \mathbb{P}^2 \dashrightarrow \tilde{S}$ . Also, by Proposition 9 this is again equivalent to asking whether or not  $\pi \circ \rho^{-1}$  is  $G_d$ -equivariant, where  $\pi : \tilde{S} \rightarrow S$  denotes the map to the associated singular del Pezzo surface. As  $r \leq 6$  however, we see that  $S \subset \mathbb{P}^{9-r}$  and moreover the map  $\pi \circ \rho^{-1}$  is given by choosing a basis for some linear series  $V \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ . We may therefore appeal to Lemma 6, and reduce to determining whether or not  $V$  is invariant under the action of  $G_d$  on  $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ .

We now show this method in action by considering the extremal surfaces given in Figure 1, beginning with the one such surface of degree five.

**Lemma 15.** *The quintic del Pezzo surface of type  $\mathbf{A}_4$  is an equivariant compactification of a homogeneous space for  $G_1$ , but is not an equivariant compactification of  $G_d$  for any  $d$ .*

*Proof.* The quintic type  $\mathbf{A}_4$  is defined by the equations

$$\begin{aligned} x_2x_4 - x_1^2 &= x_3x_4 - x_0x_1 = x_0x_2 - x_1x_3 \\ &= x_1x_2 + x_0^2 + x_4x_5 = x_2^2 + x_0x_3 + x_1x_5 = 0. \end{aligned}$$

The associated rational map from  $\mathbb{P}^2$  is given by

$$(x : y : z) \mapsto (xz^2 : yz^2 : y^2z : xyz : z^3 : -(y^3 + x^2z)).$$

This is not defined at  $(1 : 0 : 0)$ , and moreover the line  $\{z = 0\}$  is mapped to the singularity  $(0 : 0 : 0 : 0 : 0 : 1)$ . Therefore the associated action on  $\mathbb{P}^2$  must leave these subvarieties invariant, hence is of the form given in Lemma 12. For it to be equivariant, the associated linear series of cubic forms must be invariant of the action of  $G_d$ , by Lemma 6. One can check that this happens if and only if  $2k_1 = 3k_2$  (this condition comes from the term  $-(y^3 + x^2z)$ ). So for some  $k \neq 0$ , we have  $(k_1, k_2) = (3k, 2k)$ . If two of  $\alpha_1, \alpha_2, \alpha_3$  are non-zero, this leads to  $d = k = 0$ , and the action is not generically transitive. If only  $\alpha_1 \neq 0$ , we have  $d = k$ , and the action is equivalent to  $\tau_{d, -2d}$ . If only  $\alpha_2 \neq 0$ , we have  $d = 3k$ , and the action is equivalent to  $\tau_{3k, 2k}$ . If only  $\alpha_3 \neq 0$ , we have  $d = 2k$ , and the action is equivalent to  $\tau_{2k, 3k}$ . In any case, the stabiliser of a general point has order at least two and so the action is not generically free.  $\square$

We now consider the extremal surfaces of degree four.

**Lemma 16.** *For quartic generalised del Pezzo surfaces we have the following.*

- The surface of type  $\mathbf{A}_3 + 2\mathbf{A}_1$  is an equivariant compactification of  $G_d$  for all  $d \in \mathbb{Z}$ .
- The surface of type  $\mathbf{D}_4$  admits a unique structure as an equivariant compactification of  $G_2$  (but none for other  $G_d$  with  $d \geq 0$ ).
- The surface of type  $\mathbf{A}_4$  admits a unique structure as an equivariant compactification of a homogeneous space for  $G_1$  (but none for  $G_0$ ). It is also not an equivariant compactification of  $G_d$  for any  $d$ .
- The surface of type  $\mathbf{A}_3 + \mathbf{A}_1$  admits a unique structure as an equivariant compactification of  $G_1$  (but none for other  $G_d$  with  $d \geq 0$ ).
- The surface of type  $\mathbf{A}_3$  (four lines) is not an equivariant compactification of a homogeneous space for  $G_d$  for any  $d$ .

*Proof.* Type  $\mathbf{A}_3 + 2\mathbf{A}_1$ : The surface  $S$  is defined by

$$x_0x_1 - x_2^2 = x_0^2 - x_3x_4 = 0.$$

Note that this surface is toric. For each  $d \in \mathbb{Z}$ , the action of  $G_d$  is given by the representation

$$(b, t) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b^2 & t^{2d} & 2t^db & 0 & 0 \\ b & 0 & t^d & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & t^{-1} \end{pmatrix},$$

which is easily checked to be generically free and generically transitive.

Type **D**<sub>4</sub>: The surface  $S$  can be defined by

$$x_0x_3 - x_1x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0.$$

The associated rational map from  $\mathbb{P}^2$  is given by

$$(x : y : z) \mapsto (xz^2 : z^3 : yz^2 : -z(xz + y^2) : -x(xz + y^2)).$$

The associated action on  $\mathbb{P}^2$  must therefore fix  $\{z = 0\}$  and  $(1 : 0 : 0)$ , hence has the form given by Lemma 12. By considering the term  $z(xz + y^2)$ , we see that we must have  $k_1 = 2k_2$ . Also by considering the action on the final term, we see that for the linear series to be invariant we must have  $\alpha_1 = \alpha_3 = 0$  (this is due to the appearance of the monomials  $y^3$  and  $xyz$  if  $\alpha_1$  or  $\alpha_3$  are non-zero). Therefore we also have  $k_1 = d$ . Such an action may occur only when  $d$  is even, in which case it is equivalent to  $\tau_{d, -d/2}$ . This is faithful if and only if  $|d| = 2$ . The action when  $d = 2$  may be given explicitly via the representation

$$(b, t) \mapsto \begin{pmatrix} t^2 & b & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & -b & 0 & t^2 & 0 \\ -bt^2 & -b^2 & 0 & bt^2 & t^4 \end{pmatrix}.$$

Type **A**<sub>4</sub>: The surface  $S$  can be defined by

$$x_0x_1 - x_2x_3 = x_0x_4 + x_1x_2 + x_3^2 = 0.$$

The associated rational map from  $\mathbb{P}^2$  is given by

$$(x : y : z) \mapsto (z^3 : xyz : xz^2 : yz^2 : -y(x^2 + yz)).$$

The associated action on  $\mathbb{P}^2$  must therefore fix  $\{z = 0\}, (1 : 0 : 0)$  and  $(0 : 1 : 0)$ . This implies in particular that  $\alpha_1 = 0$ . By considering the final term, we see that we must have  $2k_1 = k_2$  and  $\alpha_3 = 0$  (due to a term of the form  $x^2z$  if  $\alpha_3 \neq 0$ ). Such an action is therefore equivalent to  $\tau_{d, 2d}$  for some  $d$ . This is faithful if and only if  $|d| = 1$ , in which case the stabiliser of a generic point has order 2. Explicitly the action for  $d = 1$  is given by

$$(b, t) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & t^3 & 0 & bt^2 & 0 \\ b & 0 & t & 0 & 0 \\ 0 & 0 & 0 & t^2 & 0 \\ 0 & -2bt^3 & 0 & -b^2t^2 & t^4 \end{pmatrix}.$$

Type **A**<sub>3</sub> + **A**<sub>1</sub>: Note that we originally considered this surface in [DL10, Section 5]. The equations are given by

$$x_1x_3 - x_2^2 = x_0x_3 + x_2x_4 + x_0^2 = 0.$$

The associated rational map from  $\mathbb{P}^2$  is given by

$$(x : y : z) \mapsto (xyz : y^3 : y^2z : yz^2 : -xz(x + z)).$$

The action on  $\mathbb{P}^2$  must fix  $\{y = 0\}, \{z = 0\}$  and  $(0 : 0 : 1)$ . Hence we must have  $\alpha_2 = \alpha_3 = 0$ . The linear series is invariant if and only if  $k = -d$ , in which case this action is equivalent to  $\tau_{d, d}$ . This is faithful if and only if  $|d| = 1$  and the action in the case  $d = -1$  is given in [DL10, Section 5].



Type  $\mathbf{A}_3$ : This is given by the equations

$$x_0x_1 - x_2^2 = (x_0 + x_1 + x_3)x_3 - x_2x_4 = 0.$$

It is described in [Der06a, Section 6.4]. The associated rational map from  $\mathbb{P}^2$  is given by

$$(x : y : z) \mapsto (z^3 : x^2z : xz^2 : xyz - z^3 : (x + y)(xy - z^2)).$$

The action on  $\mathbb{P}^2$  must therefore fix  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ ,  $(1 : -1 : 0)$  and the lines  $\{x = 0\}$ ,  $\{z = 0\}$ . Using Lemma 12, we must have  $\alpha_1 = \alpha_2 = 0$  and moreover  $k_1 = k_2 = d$ , as there are three fixed points. Considering the term  $(x + y)(xy - z^2)$ , we deduce that the linear series is invariant only if  $d = 0$ , which does not give a generically transitive action.  $\square$

Finally, we consider the cubic surfaces.

**Lemma 17.** *For cubic generalised del Pezzo surfaces we have the following.*

- The surface of type  $\mathbf{E}_6$  admits a unique structure as an equivariant compactification of a homogeneous space for  $G_2$  (but none for  $G_0$  or  $G_1$ ).
- The surface of type  $\mathbf{A}_5 + \mathbf{A}_1$  admits a unique structure as an equivariant compactification of a homogeneous space for  $G_1$  (but none for  $G_0$ ).

Moreover given any generically transitive action of  $G_d$  on these surfaces, any fixed point that lies on a  $(-1)$ -curve must also lie on a  $(-2)$ -curve.

*Proof.* Type  $\mathbf{E}_6$ : This is defined by  $x_3x_0^2 - x_0x_2^2 + x_1^3 = 0$ . It is the closure of the image of  $\mathbb{P}^2$  under the rational map

$$(x : y : z) \mapsto (z^3 : yz^2 : xz^2 : x^2z - y^3),$$

with  $(1 : 0 : 0)$  and  $\{z = 0\}$  in  $\mathbb{P}^2$  fixed. The only questionable part of the linear series is  $x^2z - y^3$ , which maps to an element of the linear series under the matrix in Lemma 12 if and only if  $2k_1 = 3k_2$  and  $\alpha_1 = \alpha_3 = 0$ . So there is an integer  $k$  with  $(k_1, k_2) = (3k, 2k)$ . This gives an action that is equivalent to  $\tau_{2k, 3k}$ , which is faithful if and only if  $|k| = 1$ . In this case, the stabiliser of a general point has order 3, so the action is not generically free. The induced action on the surface in the case  $k = 1$  is given by

$$(b, t) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ b & 0 & t^3 & 0 \\ b^2 & 2bt^3 & 0 & t^6 \end{pmatrix}.$$

On the only line  $\{x_0 = x_1 = 0\}$ , the only fixed point is the singularity  $(0 : 0 : 0 : 1)$ .

Type  $\mathbf{A}_5 + \mathbf{A}_1$ : This surface has equations  $x_1^3 + x_2x_3^2 + x_0x_1x_2 = 0$ . The determination of all actions of any  $G_d$  on this surface was given in [BDLB12, Lemma 4], but we reprove this result for completeness. The associated rational map is

$$(x : y : z) \mapsto (-z^3 - x^2y : yz^2 : y^2z : xyz).$$

An associated action of  $G_d$  on  $\mathbb{P}^2$  must fix the points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  and the lines  $\{y = 0\}$ ,  $\{z = 0\}$ . In the form of Lemma 12, we must have

$\alpha_1 = \alpha_3 = 0$  and  $k_1 = d$ . The associated linear series is invariant if and only if  $2k_1 = k_2$ , in which case we obtain an action on  $\mathbb{P}^2$  that is equivalent to  $\tau_{d,-2d}$ . This action is faithful if and only if  $|d| = 1$ , in which case the stabiliser of a general point has order 2. The induced action on  $S$  in the case  $d = 1$  is given by

$$(b, t) \mapsto \begin{pmatrix} t^4 & -b^2t^2 & 0 & -2bt^3 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & bt^2 & 0 & t^3 \end{pmatrix}.$$

On the only lines  $\{x_1 = x_2 = 0\}$  and  $\{x_1 = x_3 = 0\}$ , the fixed points are the singularities  $(0 : 0 : 1 : 0)$  and  $(1 : 0 : 0 : 0)$ .  $\square$

*Proof of Theorem 1.* By Lemma 16, the quartic generalised del Pezzo surfaces of types  $\mathbf{A}_3 + 2\mathbf{A}_1, \mathbf{D}_4$  and  $\mathbf{A}_3 + \mathbf{A}_1$  are equivariant compactifications of  $G_d$  for some  $d$ . Therefore, all surfaces below them in Figure 1 also are by Lemma 8. This is exactly the first collection of surfaces given in the statement of Theorem 1.

Next, by Lemma 17 we see that the cubic surfaces of types  $\mathbf{E}_6$  and  $\mathbf{A}_5 + \mathbf{A}_1$  are equivariant compactifications of homogeneous spaces for  $G_d$  for some  $d$ . Again by Lemma 8, we deduce that all surfaces below them in Figure 1 are also equivariant compactifications of homogeneous spaces for  $G_d$  for some  $d$ . Also, by Lemma 15 and Lemma 16, we know that the quintic generalised del Pezzo surface of type  $\mathbf{A}_4$  and the quartic generalised del Pezzo surface of type  $\mathbf{A}_4$  are not equivariant compactifications of  $G_d$  for any  $d$ . In particular, this implies the same result for every surface lying above them in Figure 1 by Lemma 8.

To complete the proof of Theorem 1, it suffices to show that the remaining surfaces in Figure 1 are not equivariant compactifications of homogeneous spaces for  $G_d$  for any  $d$ . For the quartic surface of type  $\mathbf{A}_3$  with four lines, this follows from Lemma 16. The cubic del Pezzo surfaces of types  $\mathbf{D}_5$  and  $\mathbf{A}_5$  have one-dimensional automorphism groups by [Sak10, Table 3], so they cannot have a generically transitive action of any  $G_d$ . Surfaces of type  $\mathbf{E}_7$  and  $\mathbf{A}_7$  of degree 2 are blow-ups of the cubic surfaces of type  $\mathbf{E}_6$  and  $\mathbf{A}_5 + \mathbf{A}_1$  in a point on one of the  $(-1)$ -curves outside the  $(-2)$ -curves. However by Lemma 17, there are no generically transitive actions fixing such points and hence these surfaces of degree 2 cannot have such an action. Finally, surfaces of type  $\mathbf{D}_6 + \mathbf{A}_1$  and  $\mathbf{D}_6$  in degree 2 and type  $\mathbf{E}_8$  in degree 1 are blow-ups of surfaces that have no generically transitive action of  $G_d$ , so they also cannot have such an action. This completes the proof of Theorem 1.  $\square$

## REFERENCES

- [AHL12] I. Arzhantsev, J. Hausen, E. Herppich, and A. Liendo. The automorphism group of a variety with torus action of complexity one, arXiv:1202.4568, 2012.
- [BDLB12] S. Baier, U. Derenthal, and P. Le Boudec. Quadratic congruences on average and rational points on cubic surfaces, arXiv:1205.0373, 2012.
- [BM90] V. V. Batyrev and Yu. I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.*, 286(1-3):27–43, 1990.
- [Bor91] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

- [Bro09] T. D. Browning. *Quantitative arithmetic of projective varieties*, volume 277 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2009.
- [BT98a] V. V. Batyrev and Yu. Tschinkel. Manin’s conjecture for toric varieties. *J. Algebraic Geom.*, 7(1):15–53, 1998.
- [BT98b] V. V. Batyrev and Yu. Tschinkel. Tamagawa numbers of polarized algebraic varieties. *Astérisque*, (251):299–340, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).
- [BW79] J. W. Bruce and C. T. C. Wall. On the classification of cubic surfaces. *J. London Math. Soc. (2)*, 19(2):245–256, 1979.
- [CLT02] A. Chambert-Loir and Yu. Tschinkel. On the distribution of points of bounded height on equivariant compactifications of vector groups. *Invent. Math.*, 148(2):421–452, 2002.
- [CT88] D. F. Coray and M. A. Tsfasman. Arithmetic on singular Del Pezzo surfaces. *Proc. London Math. Soc. (3)*, 57(1):25–87, 1988.
- [Der06a] U. Derenthal. *Geometry of universal torsors (dissertation)*. Universität Göttingen, 2006.
- [Der06b] U. Derenthal. Singular Del Pezzo surfaces whose universal torsors are hyper-surfaces, arXiv:math.AG/0604194, 2006.
- [DL10] U. Derenthal and D. Loughran. Singular del Pezzo surfaces that are equivariant compactifications. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 377(Issledovaniya po Teorii Chisel. 10):26–43, 241, 2010.
- [Dol03] I. Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
- [DP80] M. Demazure and H. C. Pinkham, editors. *Séminaire sur les Singularités des Surfaces*, volume 777 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. Held at the Centre de Mathématiques de l’École Polytechnique, Palaiseau, 1976–1977.
- [Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.
- [Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [HT99] B. Hassett and Yu. Tschinkel. Geometry of equivariant compactifications of  $\mathbf{G}_a^n$ . *Internat. Math. Res. Notices*, (22):1211–1230, 1999.
- [Man86] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.
- [Sak10] Y. Sakamaki. Automorphism groups on normal singular cubic surfaces with no parameters. *Trans. Amer. Math. Soc.*, 362(5):2641–2666, 2010.
- [TT12] S. Tanimoto and Yu. Tschinkel. Height zeta functions of equivariant compactifications of semi-direct products of algebraic groups. In *Zeta functions in algebra and geometry*, volume 566 of *Contemp. Math.*, pages 119–157. Amer. Math. Soc., Providence, RI, 2012.
- [Ye02] Q. Ye. On Gorenstein log del Pezzo surfaces. *Japan. J. Math. (N.S.)*, 28(1):87–136, 2002.

MATHEMATISCHES INSTITUT, LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN, THERE-  
SIENSTR. 39, 80333 MÜNCHEN, GERMANY

*E-mail address:* `ulrich.derenthal@mathematik.uni-muenchen.de`

DEPARTMENT OF MATHEMATICS, UNIVERSITY WALK, BRISTOL, UK, BS8 1TW

*E-mail address:* `daniel.loughran@bristol.ac.uk`